The Generalized Harry Dym Equation.

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Abstract

The Harry Dym equation is generalized to the system of equations in the same manner as the Korteweg - de Vries equation is generalized to the Hirota - Satsuma equation. The Lax and Hamiltonian formulation for this generalization is given. This generalized Lax operator gives the hierarchy of equations also.

Introduction.

Large classes of nonlinear partial differential equations are integrable by the inverse spectral transform method and its modifications [1,2]. It is possible to construct such equations using the so called Lax representation

$$L := \sum_i u_i(x)\partial^i, \tag{1}$$

$$\frac{\partial L}{\partial t} = \Big[P_{\geq k}(L^q), L\Big], \tag{2}$$

where $k = 0, 1, 2$ and q may assume suitable integer or rational values. By $P_{\geq k}$ we understand the projection to the sub-algebra of the pseudo differential symbols as discussed by Adler [3]

$$P_{\geq k} = \{\sum_{i=k}^{\infty} a(x)_i \partial^i\}. \tag{3}$$

The case $k = 2$ corresponds to the Harry Dym hierarchy [1]. It is a huge class of equations. Therefore the problem to find all or some particular reduction of this hierarchy seems to be very important.

The Harry Dym equation has been recently generalized in different ways. Antonowicz and Fordy [4] introduced and investigated the coupled Harry Dym equations and showed that these equations are connected with the new isospectral flows. Different generalization of the Harry Dym equation have been constructed in [5] as the bosonic sector of the fully supersymmetric $N = 2$ extension of the Harry Dym hierarchy. The deformed Harry Dym equation has been considered recently in [6,7,8].

Beside these extensions we would like to present new generalization of this hierarchy. We generalize Harry Dym equation in the same manner as the Korteweg de Vries equation is extended to the Hirota-Satsuma equation. This method could be considered as the admissible reduction of the Harry Dym hierarchy.

The paper is organized as follows. In the first section, we describe the Lax representation of the Hirota-Satsuma hierarchy and present its modified version. The second section, contains our basic construction where the Lax representation for the generalized Harry Dym equation is studied. In the next section, we show how it is possible to construct the hierarchy for our generalization. In the fourth section, we show that, due to the constraints putting on the Lax operator for the modified Hirota-Satsuma equation, it is impossible to construct the reciprocal link to the Lax operator for the generalized Harry Dym equation. The last section, contains concluding remarks.

1. Hirota-Satsuma equation and its modification.

The Hirota - Satsuma equation [9]

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Big(- u_{xxx} + 3v_{xxx} - 6u_x u + 6vu_x + 12v_x u\Big), \\ \frac{\partial v}{\partial t} &= \Big(- v_{xxx} + 3u_{xxx} - 6v_x v + 6v_x u + 12vu_x\Big), \end{aligned} \tag{4}$$

has the following Lax representation [10]

$$\frac{\partial L}{\partial t} = 8\Big[(L^{3/4})_{\geq 0}, L\Big], \tag{5}$$

$$L = \Big(\partial^2 + u\Big)\Big(\partial^2 + v\Big), \tag{6}$$

where ≥ 0 denotes purely differential part of $L^{3/4}$.

This Lax operator could be considered as the admissible reduction of the fourth-order Gelfand - Dikii Lax operator

$$L := f_4\partial^4 + f_3\partial^3 + f_2\partial^2 + f_1\partial + f_0, \tag{7}$$

where

$$f_4 = 1, \quad f_3 = 0, \quad f_2 = u + v, \quad f_1 = 2v_x, \quad f_0 = v_{xx} + vu. \tag{8}$$

It is possible to construct the hierarchy of system of equations using the Hirota-Satsuma Lax operator

$$\frac{\partial L}{\partial t_n} = 8\Big[(L^{(2n+1)/4})_{\geq 0}, L\Big], \tag{9}$$

where $n = 0, 1, 2....$

Let us now apply the Miura transformation to u, v

$$u = -(g_x + g^2), \quad v = -(f_x + f^2) \tag{10}$$

where f and g are the functions of x. One can easily check that, if f and g satisfy the following system of equations

$$\begin{aligned} \frac{\partial f}{\partial t} &= \Big(- f_{xx} + 2f^3 + 3g_{xx} - 6g_x f + 6g_x g - 6g^2 f \Big)_x, \\ \frac{\partial g}{\partial t} &= \Big(3f_{xx} + 6f_x f - g_{xx} + 2g^3 - 6g f_x - 6g f^2 \Big)_x, \end{aligned} \tag{11}$$

then u and v satsify the Hirota-Satsuma equation. Therefore the system of equations (11) could be considered as the modified Hirota-Satsuma equation which possesses the following Lax representation

$$\begin{aligned} \hat{L} &= e^{(-\int dx\ f)} L e^{(\int dx\ f)} = \partial^4 + 4f\partial^3 + \\ & (5f_x + 5f^2 - g_x - g^2)\partial^2 + (2f_{xx} + 6f_x f + 2f^3 - 2g_x f - 2g^2 f)\partial, \end{aligned} \tag{12}$$

$$\frac{\partial \hat{L}}{\partial t} = 8\Big[(\hat{L}^{3/4})_{\geq 1}, \hat{L}\Big]. \tag{13}$$

2. Generalized Harry Dym equation.

The Harry Dym equation [1]

$$\frac{\partial \omega}{\partial t} = \omega^3 \omega_{xxx} \tag{14}$$

can be obtained from the following Lax representation

$$L = \omega^2 \partial^2, \tag{15}$$

$$\frac{\partial L}{\partial t} = 4\Big[(L^{3/2})_{\geq 2}, L\Big]. \tag{16}$$

Let us now consider product of two different Lax operator (15) and define new Lax operator by

$$L := w^2 \partial^2 u^2 \partial^2. \tag{17}$$

Now the Lax representation

$$\frac{\partial L}{\partial t} = 4\Big[(L^{3/4})_{\geq 2}, L\Big], \tag{18}$$

leads to the system of equations on w and u

$$\begin{aligned} \frac{\partial w}{\partial t} &= w^3 \Big(w^{-1/2} u^{3/2} \Big)_{xxx}, \\ \frac{\partial u}{\partial t} &= u^3 \Big(u^{-1/2} w^{3/2} \Big)_{xxx}. \end{aligned} \tag{19}$$

It is our generalized Harry Dym equation which can be written down in the Hamiltonian form as

$$\begin{pmatrix} w \\ u \end{pmatrix}_t = J \begin{pmatrix} \frac{\delta H}{\delta w} \\ \frac{\delta H}{\delta u} \end{pmatrix} = \begin{pmatrix} 0 & w^3\partial^3 u^3 \\ u^3\partial^3 w^3 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta w} \\ \frac{\delta H}{\delta u} \end{pmatrix}, \tag{20}$$

where

$$H := -2\Big(wu\Big)^{-\frac{1}{2}}. \tag{21}$$

Let us notice that our generalized Harry Dym equation (20) can be rewritten as

$$\begin{pmatrix} f \\ g \end{pmatrix}_t = \hat{J} \begin{pmatrix} \frac{\delta H}{\delta w} \\ \frac{\delta H}{\delta u} \end{pmatrix} = \begin{pmatrix} 0 & \partial^3 \\ \partial^3 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta f} \\ \frac{\delta H}{\delta g} \end{pmatrix} = \begin{pmatrix} f^{1/4} g^{-3/4} \\ g^{1/4} f^{-3/4} \end{pmatrix}_{xxx}, \tag{22}$$

where now $H = 4(fg)^{1/4}$ and we transformed the variables w and u to

$$w \to f^{-\frac{1}{2}}, \qquad u \to g^{-\frac{1}{2}}, \tag{23}$$

The Hamiltonian operator $\hat{J}$ obviously satisfy the Jacobi identity. The transformation (23) could be considered as the Miura transformation between J and $\hat{J}$ operators. Due to it the Hamiltonian operator J satisfy the Jacobi identity too.

3. Generalized Harry Dym hierarchy.

The conserved charges for our generalized Harry Dym equations could be constructed as follows

$$\begin{aligned} H_1 \quad &= \int dx \ trL^{1/4} = \\ &- \int dx (5w_{xx}w^{-1/2}u^{1/2} + 2w_x u_x w^{-1/2}u^{-1/2} + 5u_{xx}u^{-1/2}w^{1/2})/16, \end{aligned} \tag{24}$$

$$\begin{aligned} H_2 &= \int dx \ trL^{3/4} = \int dx \\ &\Big(\Big(112 w_{xx}^2 w^{-1/2}u^{3/2} - 56 w_{xx} w_x^2 w^{-3/2} u^{3/2} - 32 w_{xx} w^{1/2} u_x^2 u^{-1/2} + 7 w_x^4 w^{-5/2} u^{3/2} \\ &\quad -144 w_{xx} w^{1/2} u_{xx} u^{1/2} - 15 w_x^2 w^{-1/2} u_x^2 u^{-1/2}\Big) + \Big(w \rightleftharpoons u\Big)\Big). \end{aligned} \tag{25}$$

Similarly to the Hirota - Satsuma hierarchy our Lax operator (17) could be considered as the admissible reduction of the fourth-order Gelfand - Dikii Lax operator where now

$$f_4 = w^2u^2, \quad f_3 = 2w^2(u^2)_x, \quad f_2 = w^2(u^2)_{xx}, \quad f_1 = 0, \quad f_0 = 0. \tag{26}$$

We can now construct the generalized hierarchy of Harry Dym equations considering the following Lax pair representation:

$$\frac{\partial L}{\partial t_n} = 4\Big[(L^{(2n+1)/4})_{\geq 2}, L\Big], \tag{27}$$

where $n = 0, 1, 2, ...$

The third member of hierarchy is

$$\begin{aligned}
\frac{\partial w}{\partial t_3} &= w^3\Big(-12w_{xx}w^{-1/2}u^{5/2} + w_x^2 w^{-3/2}u^{5/2} + \\
&\qquad 10w_x w^{-1/2}u_x u^{3/2} + 20w^{1/2}u_{xx}u^{3/2} - 15w^{1/2}u_x^2 u^{1/2}\Big)_{xxx}, \\
\frac{\partial u}{\partial t_3} &= u^3\Big(-12u_{xx}u^{-1/2}w^{5/2} + u_x^2 u^{-3/2}w^{5/2} + \\
&\qquad 10u_x u^{-1/2}w_x w^{3/2} + 20u^{1/2}w_{xx}w^{3/2} - 15u^{1/2}w_x^2 w^{1/2}\Big)_{xxx}.
\end{aligned} \tag{28}$$

This system of equations is Hamiltonian also with the same Hamiltonian operator J as in (20). We can rewrtie these equations as

$$\begin{pmatrix} w \\ u \end{pmatrix}_{t_3} = \begin{pmatrix} 0 & w^3\partial^3 u^3 \\ u^3\partial^3 w^3 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H_1}{\delta w} \\ \frac{\delta H_1}{\delta u} \end{pmatrix}. \tag{29}$$

4. A reciprocal link.

It is well known that the Harry Dym equation is connected with the Korteweg de Vries by the so called reciprocal auto-Bäcklund transformation [11]. This transformation is realised in two steps. In the first step, the Korteweg de Vries equation is transformed to the modified Korteweg de Vries equation. In the next, one performs the transformation

$$\partial \Rightarrow \Phi_x \partial' \Rightarrow \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}, \qquad x' \Rightarrow \Phi, \tag{30}$$

for the Lax operator of the modified Korteweg de Vries equation.

Let us presents how the second step works for the Harry Dym equation. If we apply the transformation to the Lax operator responsible for the modified Korteweg de Vries equation

$$L = \partial^2 + 2v\partial, \tag{31}$$

we obtain

$$L' = \Phi_x^2 \partial_{x'x'} + (\Phi_{xx} + 2\Phi_x v)\partial_{x'}, \tag{32}$$

This operator is the Lax operator for the Harry Dym equation if

$$w = \Phi_x, \quad v = -\frac{\Phi_{xx}}{2\Phi_x} \tag{33}$$

It with the transformation (30) is desired reciprocal link to the Harry Dym Lax operator.

Let us now try to apply the same strategy to our generalized Harry Dym equation. We have seen, in the first section, that the modified Hirota-Satsuma is described by tha Lax operator (12). If we use the x transformation (30) to this operator we obtain

$$\begin{aligned}
L' &= \Phi_x^4\partial'^4 + \Big(6\Phi_{xx}\Phi_x^2 + 4\Phi_x^2 f\Big)\partial'^3 \\
&\quad \Big(4\Phi_{xxx}\Phi_x + 3\Phi_{xx}^2 + 12\Phi_{xx}\Phi_x + \Phi_x^2(5f_x + 5f^2 - g_x - g^2)\Big)\partial'^2 \\
&\quad \Big(\Phi_{xxxx} + 4\Phi_{xxx}f + \Phi_{xx}(5f_x + 5f^2 - g_x - g^2) + \\
&\quad \Phi_x(2f_{xx} + 6f_x f + 2f^3 - 2g_x f - 2g^2 f)\Big)\partial'.
\end{aligned} \tag{34}$$

We immediately realize that this operator can not coincides with the Lax operator (17) of the generalized Harry Dym equation. Indeed if these two operator are the same then we should assume that $\Phi_x = w^{1/2}u^{1/2}$. Then from the second term of (34) we can fix f as a function of w and u. From the the third terms of (34) we can fix $g_x + g^2$ and substitute to the last terms in (34). However this last term does not vanish and therefore we can not construct the reciprocal link to the Lax operator of the generalized Harry Dym eqaution.

This is not in contradiction to the general construction of the reciprocal auto-Bäcklund transformation for an arbitrary Gelfand-Dikii operator described in [11]. This general construction concerns to the unconstrained operators only. The Lax operator (12) does not belong to this class. It contains two arbitrary functions but not three. These constraints does not allows to build the reciprocal transformation.

Conclusion.

In this paper we generalized the Harry Dym equation to the system of equations in the same manner as the Korteweg - de Vries equation is generalized to the Hirota - Satsuma equation. We presented the Lax and Hamiltonian formulation for this generalization. Moreover we showed that our generalization could be extended to the whole hierarchy of equations. We showed also that it is impossible to construct the reciprocal link for our Lax operator with the Lax operator of modified Hirota - Satsuma Lax operator.

Interestingly, the process of the generalizations of Harry Dym Lax operator to the product of several different Harry Dym operators, is restricted to two operators only. This can be easily verified by assuming the most general form on the solutions and checking the consistence of the Lax pair representation. The same observation concerns to the Korteveg de Vries operators.

The system of equations considered in this paper seems to be new and needs further investigations. For example the problem of construction of the bihamiltonian structure and recursion operator seems to be very tempting.

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